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Unconstrained Minimization Approaches to Nonlinear Complementarity Problems

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Abstract. The nonlinear complementarity problem can be reformulated as unconstrained minimization problems by introducing merit functions. Under some assumptions, the solution set of the nonlinear complementarity problem coincides with the set of local minima of the corresponding minimization problem. These results were presented by Mangasarian and Solodov, Yamashita and Fukushima, and Geiger and Kanzow. In this note, we generalize some results of Mangasarian and Solodov, Yamashita and Fukushima, and Geiger and Kanzow to the case where the considered function is only directionally differentiable. Some results are strengthened in the smooth case. For example, it is shown that the strong monotonicity condition can be replaced by the P-uniform property for ensuring a stationary point of the reformulated unconstrained minimization problems to be a solution of the nonlinear complementarity problem. We also present a descent algorithm for solving the nonlinear complementarity problem in the smooth case. Any accumulation point generated by this algorithm is proved to be a solution of the nonlinear complementarity under the monotonicity condition.

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1. Introduction

Let $F : \mathcal{R}^n \to \mathcal{R}^n$ be locally Lipschitzian, i.e., for any $x \in \mathcal{R}^n$, there exist a neighborhood $N(x)$ of x and a positive constant $L(x)$ such that for any $y, z \in N(x)$

$$\|F(y) - F(z)\| \leq L(x)\|y - z\|,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Given a locally Lipschitzian function F, consider the nonsmooth nonlinear complementarity problem, denoted by the NCP, which can be defined as

$$\text{find } x \in \mathcal{R}^n \text{ satisfying } x^T F(x) = 0, x \geq 0, F(x) \geq 0. \tag{1}$$

The nonlinear complementarity problem has been served as a general framework for linear, quadratical and nonlinear programming, linear complementarity problems as well as some equilibrium problems. How to design good algorithms for solving the NCP has been an active research area. See [13] for an extensive review.

When F is smooth, Mangasarian and Solodov [11] converted the NCP into an unconstrained smooth (or continuously differentiable) minimization problem by introducing a merit function $M_\alpha : \mathcal{R}^n \to \mathcal{R}$ defined by

$$\begin{aligned} M_\alpha(x) &= x^T F(x) + \frac{1}{2\alpha}[\|(-\alpha F(x) + x)_+\|^2 - \|x\|^2 \\ &\quad + \|(-\alpha x + F(x))_+\|^2 - \|F(x)\|^2], \end{aligned} \tag{2}$$

where $\alpha > 0$ is a real number and $(z)_+$ denotes the vector with components $\max\{0, z_i\}, i = 1, 2, \ldots, n$.

Kanzow [9] introduced many more so-called NCP-functions to reformulate the NCP as unconstrained minimization problems. One of them was further studied by Geiger and Kanzow [6]. This function was first introduced by Fischer [3] for reformulating nonlinear programming as a system of nonsmooth equations. We restate it as follows.

Let $\phi : \mathcal{R}^2 \to \mathcal{R}$ be a function defined by

$$\phi(a, b) = \frac{1}{2}(\sqrt{a^2 + b^2} - a - b)^2. \tag{3}$$

The Geiger and Kanzow's merit function $Q : \mathcal{R}^n \to \mathcal{R}$ can be defined by

$$Q(x) = \sum_{i=1}^{n} \phi(x_i, F_i(x)). \tag{4}$$

For these two merit functions, we state some fine properties taken from [11] and [6] in the following proposition.

PROPOSITION 1. *Suppose F is smooth on $\mathcal{R}^n$ and $\alpha > 1$. Let $P = M_\alpha$ or Q. Then*

(i) $P(x) \geq 0$ for any $x \in \mathcal{R}^n$,

(ii) P is smooth on $\mathcal{R}^n$,

(iii) $P(x) = 0$ if and only if x solves the NCP,

(iv) the set of solutions of the NCP coincides with the set of global minima of P if the NCP has a solution.

The above results show that one may solve the NCP by finding the *global* minima of the following unconstrained smooth minimization problems

$$\min_{x \in \mathcal{R}^n} M_\alpha(x), \tag{5}$$

or

$$\min_{x \in \mathcal{R}^n} Q(x). \tag{6}$$

It is well known that a local minimum is not generally a global minimum for an unconstrained minimization problem. And the problem of finding a global minimum is generally difficult because most unconstrained minimization methods normally generate a sequence converging to a local minimizer or a stationary point rather than a global minimizer. It is therefore crucial to study under what conditions a stationary point of unconstrained minimization problems becomes a solution of the NCP. Yamashita and Fukushima [14] presented a sufficient condition to assure that the set of stationary points of (5) does coincide with the set of solutions of the NCP. This is stated in the proposition below.

PROPOSITION 2. *Suppose F is smooth on $\mathcal{R}^n$ and $\alpha > 1$. If the Jacobian $\nabla F(x)$ is positive definite for all $x \in \mathcal{R}^n$, then any stationary point of M_α is a solution of the NCP.*

Following the same line, Geiger and Kanzow gave a similar result for the merit function Q under a weaker condition.

PROPOSITION 3. *Suppose F is smooth on $\mathcal{R}^n$. If F is monotone on $\mathcal{R}^n$, then any stationary point of Q is a solution of the NCP.*

Furthermore, instead solving (5) and (6) directly by invoking minimization methods, Yamashita and Fukushima [14], and Geiger and Kanzow [6] proposed a descent algorithm respectively, which does not require the derivative information of F, and M_α or Q, for solving (5) and (6) when F is smooth and strongly monotone on $\mathcal{R}^n$.

In this note, we generalize Propositions 1, 2 and 3 to the case where F is only directionally differentiable in Section 2. When F is smooth, some weaker or different conditions are established to assure that any stationary point of the merit function is a solution of the NCP. It is shown that the P-uniform property of the function F is sufficient for a stationary point of the unconstrained minimization problems (5) and (6) to be a solution of the NCP. Section 3 is devoted to the compactness of the level sets of merit functions, which is crucial in establishing the convergence of some minimization algorithms for solving the NCP. We show the compactness of the level sets of the two merit functions by weakening not only the condition of the smoothness of F but also the strong monotonicity of F. In Section 4, we present a descent method by using the descent direction of Q which was introduced by Geiger and Kanzow [6], for solving the NCP. This method also does not need to calculate the derivatives of F and Q. Under the condition that F is monotone, any accumulation point generated by this method is shown to be a solution of the NCP. We remark that the strong monotonicity condition is required for establishing the convergence of the descent method proposed by Geiger and Kanzow [6].

2. The Equivalence Results

F is said to be directionally differentiable at x if for any $d \in \mathcal{R}^n$, the following limit exists

$$\lim_{t \to 0+} \frac{F(x+td) - F(x)}{t}.$$

The limit is denoted by $F'(x, d)$. Furthermore, F is called directionally differentiable on a set if it is directionally differentiable at any point of the given set. The next proposition is a generalization of Proposition 1 for the merit function M_α.

PROPOSITION 4. *Suppose F is directionally differentiable on $\mathcal{R}^n$. Then for any $\alpha > 1$,*

(i) $M_\alpha(x) \geq 0$ for any $x \in \mathcal{R}^n$,

(ii) M_α is directionally differentiable on $\mathcal{R}^n$,

(iii) $M_\alpha(x) = 0$ if and only if x solves the NCP,

(iv) the set of solutions of the NCP coincides with the set of global minima of M_α if the NCP has a solution.

Proof. We only show (ii). The other statements follow a similar argument given by Mangasarian and Solodov [11] and Yamashita and Fukushima [14]. by the definition of directional differentiability, for any $x, d \in \mathcal{R}^n, t > 0$,

$$\begin{aligned}
M_\alpha(x+td) - M_\alpha(x) &= ((x+td)^T F(x+td) - x^T F(x)) \\
&+ \frac{1}{2\alpha}[(\|(-\alpha F(x+td) + (x+td))_+\|^2 - \|(-\alpha F(x) + x)_+\|^2) \\
&-(\|x+td\|^2 - \|x\|^2) + (\|(-\alpha(x+td) + F(x+td))_+\|^2 \\
&-\|(-\alpha x + F(x))_+\|^2) - (\|F(x+td)\|^2 - \|F(x)\|^2)].
\end{aligned}$$

Dividing the above expression by t and taking the limit, by a careful calculation, we have

$$\begin{aligned}
M'_\alpha(x, d) &= x^T F'(x, d) + d^T F(x) \\
&\quad + \frac{1}{\alpha}[g(x, d) - x^T d + h(x, d) - F(x)^T F'(x, d)],
\end{aligned}$$

where

$$\begin{aligned}
g(x, d) &= \sum_{i=1}^{n} (-\alpha F_i(x) + x_i)_+ (-\alpha F'_i(x, d) + d_i), \\
h(x, d) &= \sum_{i=1}^{n} (-\alpha x_i + F_i(x)_+ (-\alpha d_i + F'_i(x, d)).
\end{aligned}$$ □

In order to generalize Proposition 2 for the merit function M_α, we summarize some related results from [14].

LEMMA 1. *Suppose that F is continuous on $\mathcal{R}^n$ and $\alpha > 0$. Then the following three statements are equivalent:*
(i) $F(x) \geq 0, x \geq 0, x^T F(x) = 0,$
(ii) $x = (-\alpha F(x) + x)_+,$
(iii) $F(x) = (-\alpha x + F(x))_+.$
Define two functions $G : \mathcal{R}^n \to \mathcal{R}^n$ and $H : \mathcal{R}^n \to \mathcal{R}^n$ by

$$G(x) = (-\alpha F(x) + x)_+ - x$$

and

$$H(x) = (-\alpha x + F(x))_+ - F(x).$$

LEMMA 2. *For any $\alpha > 0, x \in \mathcal{R}^n$ and $i \in \{1, 2, \ldots, n\}$, we have*

$$(-\alpha G_i(x) + H_i(x))^T (G_i(x) - \alpha H_i(x)) \geq 0,$$

hence,

$$(-\alpha G(x) + H(x))^T (G(x) - \alpha H(x)) \geq 0.$$

Recall some definitions from [12]. $F : \mathcal{R}^n \to \mathcal{R}^n$ is said to be a uniform P-function on a set S if there exists a positive constant λ such that for $x, y \in S$

$$\max_{1 \leq i \leq n} (x_i - y_i)(F_i(x) - F_i(y)) \geq \lambda \|x - y\|^2.$$

F is said to be strongly monotone on a set S if there exists a positive constant λ such that for $x, y \in S$

$$(x - y)^T (F(x) - F(y)) \geq \lambda \|x - y\|^2.$$

F is said to be monotone on a set S if $\lambda = 0$ in the above inequality. It is well known that the strong monotonicity is stronger than both the uniform P function property and the monotonicity.

We establish some result related to the function F.

LEMMA 3. *Suppose F is a directionally differentiable on an open set S. Then, for any $x \in S$ and $d \in \mathcal{R}^n$,*
(i) $$\max_{1 \leq i \leq n} d_i F_i'(x, d) \geq \lambda \|d\|^2,$$

if F is a uniform P-function on S with modulus λ,

(ii) $$d^T F'(x, d) \geq \lambda \|d\|^2,$$

if F is strongly monotone on S with modulus λ,

(iii) $$d^T F'(x, d) \geq 0,$$

if F is monotone on S.

Proof. It can be verified by the definitions. □

We now establish an equivalent result for the merit function M_α.

PROPOSITION 5. *Suppose F is directionally differentiable and strongly monotone on an open set S. If $x^* \in S$ is a stationary of M_α, i.e., $M'_\alpha(x^*, d) \geq 0$ for any $d \in \mathcal{R}^n$, then x^* is a solution of the NCP for any $\alpha > 1$.*

Proof. Since x^* is a stationary point of M_α, $M'_\alpha(x^*, d) \geq 0$ for any $d \in \mathcal{R}^n$. Rewrite $\alpha M'_\alpha(x, d)$ as

$$\begin{aligned}\alpha M'_\alpha(x,d) &= (\alpha x - F(x))^T F'(x,d) + d^T(\alpha F(x) - x)\\ &\quad +(G(x)+x)^T(-\alpha F'(x,d)+d)\\ &\quad +(H(x)+F(x))^T(-\alpha d + F'(x,d)),\\ &= (H(x) - \alpha G(x))^T F'(x,d) + d^T(G(x) - \alpha H(x)).\end{aligned}$$

Let $d^* = \alpha G(x^*) - H(x^*)$. Then $M'_\alpha(x^*, d^*) \geq 0$ gives

$$\begin{aligned}&-(\alpha G(x^*) - H(x^*))^T F'(x^*, d^*) + (G(x^*)\\ &-\alpha H(x^*))^T(\alpha G(x^*) - H(x^*)) \geq 0,\end{aligned}$$

i.e.,

$$(d^*)^T F'(x^*, d^*) - (G(x^*) - \alpha H(x^*))^T d^* \leq 0.$$

This implies that

$$(d^*)^T F'(x^*, d^*) \leq (G(x^*) - \alpha H(x^*))^T d^* \leq 0,$$

the second inequality is due to Lemma 2. Therefore, the strong monotonicity of F on S shows $d^* = 0$ by (ii) of Lemma 3. Consequently, for any $d \in \mathcal{R}^n$

$$M'_\alpha(x^*, d) = d^T(G(x^*) - \alpha H(x^*)) \geq 0,$$

which shows

$$G(x^*) - \alpha H(x^*) = 0.$$

In view of $d^* = \alpha G(x^*) - H(x^*) = 0$ and $\alpha > 1$, we have

$$G(x^*) = 0, H(x^*) = 0.$$

The conclusion follows from Lemma 1. □

When F is smooth on S, the above proposition recovers the main result-Theorem 2.1 in [14]. Importantly, the strong monotonicity of F on S can be weakened. We present such a result below.

PROPOSITION 6. *Suppose F is a differentiable uniform P-function on an open set S with modulus λ. If $x^* \in S$ is a stationary of M_α, then x^* is a solution of the NCP for any $\alpha > 1$.*

Proof. Since x^* is a stationary point of M_α, $\nabla M_\alpha(x^*) = 0$. Rewrite $\nabla M_\alpha(x)$ as

$$\nabla M_\alpha(x) = \nabla F(x)^T(H(x) - \alpha G(x)) + (G(x) - \alpha H(x)).$$

Then, $\nabla M_\alpha(x^*) = 0$ gives

$$\begin{gathered}(\nabla F(x^*)^T(H(x^*) - \alpha G(x^*)))_1 + (G_1(x^*) - \alpha H_1(x^*)) = 0 \\ \vdots \\ (\nabla F(x^*)^T(H(x^*) - \alpha G(x^*)))_n + (G_1(x^*) - \alpha H_n(x^*)) = 0,\end{gathered}$$

where $(\nabla F(x^*)^T(H(x^*) - \alpha G(x^*)))_i$ denotes the i-th element of the column vector $\nabla F(x^*)^T(H(x^*) - \alpha G(x^*))$. Let $d^* = -\alpha G(x^*) + H(x^*)$. Multiplying the i-th equation above by d_i^*, we obtain

$$\begin{gathered}d_i^*(\nabla F(x^*)^T d^*)_i + (-\alpha G_i(x^*) + H_i(x^*))(G_i(x^*) - \alpha H_i(x^*)) = 0, \\ i = 1, 2, \ldots, n.\end{gathered}$$

Therefore, Lemma 2 shows that

$$\max_{1 \le i \le n} d_i^*(\nabla F(x^*)^T d^*)_i \le 0,$$

which implies $d^* = 0$ by the fact that F is a uniform P-function (which implies that both $\nabla F(x^*)$ and $\nabla F(x^*)^T$ are P-matrices) and Lemma 3. Analogous to the proof of Proposition 5, one can show that x^* is a solution to the NCP. □

For the merit function Q, one can establish the same result as Proposition 4. We do not give all the detail here. However, if F is directionally differentiable on $\mathcal{R}^n$, then Q is also directionally differentiable on $\mathcal{R}^n$ and for any $x, d \in \mathcal{R}^n$,

$$Q'(x, d) = \sum_{i=1}^{n} \phi'(x_i, F_i(x), d),$$

where $\phi'(x_i, F_i(x), d) = 0$ if $x_i^2 + (F_i(x))^2 = 0$, otherwise

$$\phi'(x_i, F_i(x), d) = \nabla_a \phi(x_i, F_i(x)) d_i + \nabla_b \phi(x_i, F_i(x)) F_i'(x, d).$$

A similar result to Proposition 5 can be established. This is a generalization of Theorem 2.5 in [6]. We first state some preliminary results from [6].

LEMMA 4. *(i) $\phi(a, b) \ge 0$, for any $a, b \in \mathcal{R}$.*

(ii) $\phi(a, b) = 0$ if and only if $a \ge 0, b \ge 0, ab = 0$.

(iii) ϕ *is continuously differentiable for all* $a, b \in \mathcal{R}$, *in particular* $\nabla\phi(0,0) = (0,0)^T$.

(iv) $\nabla_a\phi(a,b)\nabla_b\phi(a,b) \geq 0$, *for any* $a, b \in \mathcal{R}$.

(v) If $\nabla_a\phi(a,b)\nabla_b\phi(a,b) = 0$, *then* $\phi(a,b) = 0$.

PROPOSITION 7. *Suppose* F *is directionally differentiable and monotone on an open set* S. *If* $x^* \in S$ *is a stationary of* Q, *then* x^* *is a solution of the NCP.*

Proof. The detail of the proof is omitted. It is analogous to that of Proposition 5. □

When F is smooth, one result similar to Proposition 6 is given by Jiang and Qi [8]. We state it below.

PROPOSITION 8. *Suppose* F *is a differentiable uniform P-function on an open set* S. *If* $x^* \in S$ *is a stationary of* Q, *then* x^* *is a solution of the NCP.*

3. The Convergence Properties

As was discussed before, it is always important to study the compactness of the level set of unconstrained minimization. Define a function $N_\alpha : \mathcal{R}^2 \to \mathcal{R}$

$$N_\alpha(a,b) = ab + \frac{1}{2\alpha}(\|(-\alpha b + a)_+\|^2 - \|a\|^2 + \|(-\alpha a + b)_+\|^2 - \|b\|^2).$$

LEMMA 5. *Let* $\alpha > 1$. *Then,* $N_\alpha(a,b)$ *is unbounded whenever* $a \to \infty, b \to \infty$.

Proof. Without loss of generality, consider the following four cases.

Case 1. $a \to +\infty$ and $b \to -\infty$. By an algebraic manipulation, for all sufficiently large a and b, we have

$$N_\alpha(a,b) = \frac{1}{2\alpha}(\alpha^2 - 1)b^2 \to +\infty.$$

Case 2. $a \to -\infty$ and $b \to +\infty$. Using the same argument as Case 1, for all sufficiently large a and b, we obtain

$$N_\alpha(a,b) = \frac{1}{2\alpha}(\alpha^2 - 1)a^2 \to +\infty.$$

Case 3. $a \to +\infty$ and $b \to +\infty$. By the definition of N_α and the fact that

$$x_+ = \max(0,x) = \frac{x + |x|}{2},$$

we have

$$\begin{aligned} N_\alpha(a,b) &= \frac{1}{4\alpha}((\alpha^2 - 1)a^2 + (\alpha^2 - 1)b^2 \\ &\quad + (-\alpha a + b)|-\alpha a + b| + (-\alpha b + a)|-\alpha b + a|) \\ &\geq \frac{1}{2\alpha}(2\alpha ab - a^2 - b^2) \\ &= \frac{ab}{2\alpha}(2\alpha - a/b - b/a). \end{aligned}$$

Suppose $\{N_\alpha(a,b)\}$ is bounded. Then, by passing to the subsequence, it follows

$$2\alpha - a/b - b/a \to 0.$$

This shows that $\{a/b\}$ and $\{b/a\}$ are bounded. By passing to the subsequence again, we may assume

$$a/b \to \alpha_1, b/a \to \alpha_2.$$

Consequently,

$$\begin{aligned}
&\alpha_1 > 0, \alpha_2 > 0,\\
&\alpha_1 + \alpha_2 = 2\alpha,\\
&\alpha_1\alpha_2 = 1,\\
&\alpha_{1,2} = \alpha \pm \sqrt{\alpha^2 - 1}.
\end{aligned}$$

We may assume $\alpha_1 > \alpha$, and $\alpha_2 < \alpha$. Hence, when a and b are sufficiently large enough,

$$a - \alpha b > 0, b - \alpha a < 0.$$

By the definition of N_α, for all sufficiently large enough a and b, we obtain

$$N_\alpha(a,b) = \frac{\alpha^2 - 1}{2\alpha} b^2 \to +\infty.$$

It is a contradiction.

Case 4. $a \to -\infty$ and $b \to -\infty$. One can prove this case as Case 3. □

PROPOSITION 9. *Suppose F is continuous on $\mathcal{R}$. Then, the level set $L(x^0) = \{x | M_\alpha(x) \leq M_\alpha(x^0)\}$ is compact if F is a uniform P-function with modulus λ.*

Proof. Suppose $\{x^k\} \subseteq L(x^0)$ and $\lim_{k\to\infty} \|x^k\| = \infty$. Define

$$J = \{1 \leq i \leq n | \{x_i^k\} \text{ is unbounded}\}.$$

Then $J \neq \emptyset$. Let

$$y_i^k = \begin{cases} 0 & i \in J, \\ x_i^k & i \notin J. \end{cases}$$

Consequently, we have

$$\begin{aligned}
\lambda \sum_{i\in J} (x_i^k)^2 &= \lambda \|x^k - y^k\|^2\\
&\leq \max_{1\leq i\leq n} (x_i^k - y_i^k)(F_i(x^k) - F_i(y^k))\\
&\leq \max_{i\in J} (x_i^k - y_i^k)(F_i(x^k) - F_i(y^k))\\
&\leq \max_{i\in J} |x_i^k||F_i(x^k) - F_i(y^k)|\\
&\leq \sqrt{\sum_{i\in J} (x_i^k)^2} \max_{i\in J} |F_i(x^k) - F_i(y^k)|.
\end{aligned}$$

This shows

$$\lambda\sqrt{\sum_{i\in J}(x_i^k)^2} \leq \max_{i\in J}|F_i(x^k) - F_i(y^k)|.$$

Then $\{|F_j(x^k)|\}$ is unbounded for some $j \in J$. Choose $l \in J$ such that both $\{x_l^k\}$ and $\{F_l(x^k)\}$ are unbounded. Then, Lemma 5 shows that $\{N_\alpha(x_l^k, F_l(x^k))\}$ is unbounded. The nonnegativeness of $N_\alpha(x_i^k, F_i(x^k))$ for $i = 1, 2, \ldots n$ implies that $\{M_\alpha(x^k)\}$ is unbounded. It is a contradiction. □

When F is strongly monotone, the above proposition recovers Theorem 2.3 in [14]. For the merit function Q, the compactness of the corresponding level set can also be established under the assumption that F is a uniform P-function. The following proposition generalizes Theorem 3.2 in [6], where F is assumed strongly monotone.

PROPOSITION 10. *Suppose F is continuous on $\mathcal{R}^n$. Then, the level set $L(x^0) = \{x|Q(x) \leq Q(x^0)\}$ is compact if F is a uniform P-function.*

Proof. Suppose $\{x^k\} \subseteq L(x^0)$ and $\lim_{k\to\infty}\|x^k\| = \infty$. As in the proof of Proposition 9, there exists $l \in \{1, 2, \ldots, n\}$ such that both $\{x_l^k\}$ and $\{F_l(x^k)\}$ are unbounded. Therefore, $\{\phi(x_l^k, F_l(x^k)\}$ is unbounded by Lemma 3.1 in [6]. It follows that $\{Q(x^k)\}$ is unbounded. This is a contradiction. □

4. The Convergence of a Descent Direction Method

Results obtained in the previous sections suggest that we may use any unconstrained minimization method for solving the NCP. When F is smooth on $\mathcal{R}^n$, however, Yamashita and Fukushima [14], Geiger and Kanzow [6] proposed a descent method for minimizing the unconstrained minimization (5) and (6) respectively, which does not require to compute the derivatives of F and M_α or Q. Two methods are proved convergent to the unique solution of the NCP under the condition that F is strongly monotone on $\mathcal{R}^n$. One may attempt to generalize these two methods to the case where F is only directionally differentiable. Many methods for solving nonsmooth unconstrained minimization have been developed in the last two decades. We refer the reader to [7]. Therefore, it is always possible to solve (5) and (6) even when F is directionally differentiable on $\mathcal{R}^n$. In this section, we present an algorithm for solving the NCP when F is smooth and monotone.

Let

$$d^k = -(\nabla_b\phi(x_1^k, F_1(x^k)), \ldots, \nabla_b\phi(x_n^k, F_n(x^k)))^T.$$

Geiger and Kanzow [6] used d^k as the search direction in their method since d^k is a descent direction if x^k is not a solution of the NCP. They proved the convergence of their method under the condition of strong monotonicity of F. However, the strong monotonicity condition is unfavourable. As we shall see, this condition can be replaced by the monotonicity condition of F. To this end, we present the modified Geiger and Kanzow's descent method as follows.

ALGORITHM 1.
Step 1. Let $x^0 \in \mathcal{R}^n, \epsilon > 0, \sigma \in (0,1)$ and $\beta \in (0,1)$. Set $k = 0$.
Step 2. If $Q(x^k) < \epsilon$, stop. Otherwise, go to Step 3.
Step 3. Find the smallest nonnegative integer, say m^k, satisfying

$$Q(x^k + \beta^{m^k} d^k) - Q(x^k) \le -\sigma(\beta^{m^k})^2 Q(x^k). \tag{7}$$

Step 4. Let $x^{k+1} = x^k + \beta^{m^k} d^k, k := k+1$ and go to Step 2.

REMARK. In Step 3, a change is made for the line search rule. This change is crucial for the convergence of Algorithm 1.

The convergence of Algorithm 1 is presented below.

PROPOSITION 11. *Suppose F is smooth and monotone on $\mathcal{R}^n$. Then, Algorithm 1 is well-defined for any initial point x^0. Furthermore, if x^* is an accumulation point of the sequence $\{x^k\}$ generated by Algorithm 1, then x^* is a solution of the NCP.*

Proof. We first prove that Algorithm 1 is well defined. Suppose x^k has been well defined by Algorithm 1. Then d^k is also well defined. Assume that there is no nonnegative integer satisfying (7). It follows that for any integer $l \ge 0$

$$Q(x^k + \beta^l d^k) - Q(x^k) > -\sigma(\beta^l)^2 Q(x^k).$$

Dividing the above inequality by β^l and letting $l \to \infty$, we have

$$Q'(x^k, d^k) \ge 0.$$

Since Q is continuously differentiable on $\mathcal{R}^n$,

$$\nabla Q(x^k)^T d^k = Q'(x^k, d^k) \ge 0.$$

However, Geiger and Kanzow [6] have shown that d^k is a descent direction of Q at x^k if x^k is not a solution of the NCP, i.e., $\nabla Q(x^k)^T d^k < 0$. It is a contradiction. Thus the well-definedness of Algorithm 1 follows. Furthermore, the objective function Q is decreased after each iteration.

Assume that x^* is an accumulation point of $\{x^k\}$, say the limit of the subsequence of $\{x^k, k \in K\}$. Then $\{x^k, k \in K\}$ is bounded, which implies the boundedness of $\{d^k, k \in K\}$ by the continuous differentiability of ϕ. Without loss of generality, we may assume $d^k \to d^*$ as $k(\in K) \to \infty$. If $\{m^k, k \in K\}$ is bounded, then Step 3 of Algorithm 1 gives

$$\sum_{k \in K} Q(x^k) < \infty.$$

This shows $Q(x^*) = 0$, i.e., x^* is a solution of the NCP by (iii) of Proposition 1. Assume that $\{m^k, k \in K\}$ is unbounded. Clearly,

$$\nabla Q(x^*)^T d^* \leq 0. \tag{8}$$

On the other hand, we aim at proving $\nabla Q(x^*)^T d^* \geq 0$. We may assume $m^k (k \in K) \to \infty$ by passing to the subsequence. It follows from the line search rule of Algorithm 1 that for $k \in K$

$$Q(x^k + \beta^{m^k} d^k) - Q(x^k) \leq -\sigma(\beta^{m^k})^2 Q(x^k)$$

and

$$Q(x^k + \beta^{m^k - 1} d^k) - Q(x^k) > -\sigma(\beta^{m^k - 1})^2 Q(x^k).$$

Dividing through the second inequality above by β^{m^k-1} and taking the limit, we have

$$\nabla Q(x^*)^T d^* = Q'(x^*, d^*) \geq 0,$$

where the right hand side follows from the boundedness of $\{Q(x^k)\}$. Consequently, by (8)

$$\nabla Q(x^*)^T d^* = 0.$$

Next, we show that x^* is a solution of the NCP. In fact, by (iii) of Lemma 4, it follows

$$\nabla Q(x^*)^T d^* = -\sum_{i=1}^{n} \nabla_a \phi(x_i^*, F_i(x^*)) \nabla_b \phi(x_i^*, F_i(x^*)) - (d^*)^T \nabla F(x^*) d^*.$$

Consequently, the monotonicity of F and (iv) of Lemma 4 imply

$$\nabla_a \phi(x_i^*, F_i(x^*)) \nabla_b \phi(x_i^*, F_i(x^*)) = 0, i = 1, 2, \ldots, n.$$

It follows from (ii) and (v) of Lemma 4 that x^* is a solution to the NCP. □

REMARK. Recently, some progress has been made concerning different properties of the merit functions M_α and Q. The interested reader is referred to [1, 2, 4, 10].

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